Transient and busy period analysis of the $GI/G/1$ queue: The method of stages

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In this paper we study the transient behavior of the $MGE_L/MGE_M/1$ queueing system, where MGE is the class of mixed generalized Erlang distributions which can approximate an arbitrary distribution. We use the method of stages combined with the separation of variables and root finding techniques together with linear and tensor algebra. We find simple closed form expressions for the Laplace transforms of the queue length distribution and the waiting time distribution under FCFS when the system is initially empty and the busy period distribution. We report computational results by inverting these expressions numerically in the time domain. Because of the simplicity of the expressions derived our algorithm is very fast and robust.

Keywords: Transient analysis; busy period; transform methods; linear algebra.

1. Introduction

Transient and busy period analysis in queueing models have long been considered as very difficult problems. Yet, in many situations it is very important to study the transient behavior of queueing systems. For example, systems often encounter transient behavior due to exogenous changes, such as the opening or closing of a queueing system or the application of a new control. Furthermore, even in systems with time homogeneous behavior the convergence to steady state is so slow that the equilibrium behavior is not indicative of system behavior. Examples from practical situations, in which transient phenomena are

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important, include manufacturing systems with frequent start up periods and transportation systems with time varying demand (for example airport runway operations in major airports).

Analytical investigations of the transient behavior of queueing systems are very rare, mainly because of the complexity involved. For the $M/M/1$ queue expressions for the queue length probabilities are known as sums of modified Bessel functions (see Gross and Harris [6]). An indication of the interest of the research community in transient behavior of queues is the recent work of Abate and Whitt [1] for the $M/M/1$ queue.

In the last decade, work on the transient behavior of queueing systems has concentrated on numerical techniques. This change in emphasis was primarily motivated by the analytical complexity of the problems involved. The two principal methods are the randomization technique introduced by Grassmann [5] and numerical integration methods of the underlying Kolmogorov differential equations (see Gross and Harris [6], section 7.3.2 and references therein).

In this paper we study various transient performance characteristics of the $MGE_L/MGE_M/1$ system, which is an important special case of the $GI/G/1$ queue. In a sequel paper (Bertsimas et al. [3]) we formulate the problem of finding simultaneously the waiting time distribution and the busy period distribution of the $GI/G/1$ queue with arbitrary distributions as a Hilbert problem. The MGE, which is described in some detail in section 2, is the class of mixed generalized Erlang distributions, which is dense in the space of all distributions and can approximate arbitrarily closely every distribution at the expense of requiring a large number of stages. For a discussion of the properties of the MGE class see Bertsimas [2]. We study the queue length, the waiting time and the busy period distribution. We use the separation of variables technique together with root finding techniques to establish closed form expressions for the Laplace transform of the distributions under study (queue length, waiting time, busy period). The advantage of these closed form expressions is that they are relatively simple and can be used for numerically inverting them in the time domain. In fact, in section 6 we report computational results for inverting numerically the transform of the busy period, the queue length and the waiting time distribution.

These expressions also explain the difficulty that the research community has had over the years in establishing expressions for the distributions we study in the time domain. Despite their simplicity in the transform domain, our expressions involve roots of polynomial equations. In general, these roots can not be computed analytically and even if they are known they are complicated enough to make their analytic inversion extremely complicated if not impossible.

The paper is structured as follows. In section 2 we describe the MGE distribution and the notation we use. In section 3 we derive closed form expressions for the transform of the queue length distributions when the system is initially empty. In section 4 we find an explicit expression for the busy period

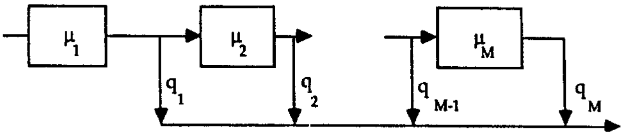

Fig. 1. The MGE_M class of distributions.

distribution while in section 5 we analyze the waiting time distribution under FCFS. In section 6 we describe the algorithm to invert numerically the closed form expressions derived in the previous sections and also report some preliminary computational results. The final section contains some concluding remarks.

2. Model formulation and notation

The general Coxian class C_n was introduced in Cox's [4] pioneering paper. It consists of a series of exponential stages as shown in fig. 1. It should be noted that this stage representation of the Coxian distribution is purely formal in the sense that the branching probabilities q_i can be negative and the rates μ_i can be complex numbers. The mixed generalized Erlang distribution (MGE) is a Coxian distribution, where we assume that the probabilities q_i are non-negative and the rates μ_i are reals. As a result, the mixed generalized Erlang distribution has a valid probabilistic interpretation, which is further exploited in this paper.

To analyse the model we conceive of the arrival process as an arrival timing channel (ATC) consisting of L consecutive exponential stages with rates λ_1, $\lambda_2, \dots, \lambda_L$ and with probabilities $p_1, p_2, \dots, p_L = 1$ of entering the system after the completion of the 1st, 2nd,$\dots$, Lth stage. We remark that as soon as a customer in the ATC enters the system a new customer arrives at stage 1 of the ATC. For the service time distribution we consider as above a service-timing channel (STC) consisting of M consecutive exponential stages with rates μ_1, $\mu_2, \dots, \mu_M$ and with probabilities $q_1, q_2, \dots, q_M = 1$ of leaving the system.

In the time domain we introduce the random variables:

$N(t)$ = The number of customers in the system at time t.
$R_a(t)$ = The ATC stage currently occupied by the arriving customer at time t.
$R_s(t)$ = The STC stage currently occupied by the customer who is being served at time t.
$W(t)$ = The waiting time of a customer arriving at time t.

With the above definitions the system can be formulated as a continuous time Markov chain with infinite state space:

$$\{(N(t), R_a(t), R_s(t)), N(t) = 0, 1, \dots, R_a(t) = 1, 2, \dots, L, \\ R_s(t) = 1, 2, \dots, M\}.$$

We now introduce the following set of probabilities:

$$P_{n,i,j}(t) = Pr\{N(t) = n,\ R_a(t) = i,\ R_s(t) = j\},$$

$$P_{0,i}(t) = Pr\{N(t) = 0,\ R_a(t) = i\}.$$

We will also use the following notation:

- $1/\lambda$ = the mean interarrival time, $1/\mu$ = the mean service time and $\rho = \lambda/\mu$ = the traffic intensity. Note that $1/\lambda = \sum_{k=1}^{L}(1/\lambda_k)p_k\prod_{r=1}^{k-1}(1-p_r)$ and $1/\mu = \sum_{k=1}^{M}(1/\mu_k)q_k\prod_{r=1}^{k-1}(1-q_r)$.
- $\boldsymbol{P}_n(t)$ = a column vector, whose elements are the probabilities $P_{n,i,j}(t)$.
- $a_k(t)$, $b_r(t)$ = the probability density function (pdf) of the remaining interarrival (service) time if the customer in the ATC (STC) is in stage $k = 1, \ldots, L$ ($r = 1, \ldots, M$). Note that because of the memoryless property of the exponential distribution $a_1(t)$, $b_1(t)$ is the pdf of the interarrival (service) time.
- $a_i^j(t)$, $b_i^j(t)$ = the probability to move from stage $i \leqslant j$ of the ATC (STC) to stage j during the interval t without having any new arrival (service completion).
- $\boldsymbol{a}_1(t) = (a_1^1(t), \ldots, a_1^L(t))'$, $\boldsymbol{a}_k(t) = (0, \ldots, a_k^k(t), \ldots, a_k^L(t))'$.
- $\boldsymbol{b}_1(t) = (b_1^1(t), \ldots, b_1^M(t))'$, $\boldsymbol{b}_k(t) = (0, \ldots, b_k^k(t), \ldots, b_k^M(t))'$.
- $\boldsymbol{\pi}_n(s)$, $\boldsymbol{\alpha}_k(s)$, $\boldsymbol{\beta}_k(s)$, $\alpha_k(s)$, $\beta_k(s)$ = the Laplace transforms of $\boldsymbol{P}_n(t)$, $\boldsymbol{a}_k(t)$, $\boldsymbol{b}_k(t)$, $a_k(t)$, $b_k(t)$ respectively.
- $\boldsymbol{e}_j = (0, \ldots, 1, \ldots, 0)'$ with an 1 at the jth coordinate.

By introducing the following upper semidiagonal matrices A_0, B_0 and the dyadic matrices A_1, B_1:

$$A_0 = \begin{bmatrix} \lambda_1 & -(1-p_1)\lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \lambda_L \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -p_1\lambda_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \\ -\lambda_L & 0 & \cdots & 0 \end{bmatrix},$$

and similarly B_0, B_1 for the service time distribution we have

$$\boldsymbol{\alpha}_k'(s) = \boldsymbol{e}_k'(Is + A_0)^{-1},$$

$$\alpha_k(s) = -\boldsymbol{e}_k(Is + A_0)^{-1}A_1\boldsymbol{e}_1 = \sum_{r=k}^{L} p_r\lambda_r\alpha_k^r(s) = \sum_{r=k}^{L} p_r\lambda_r\frac{\prod_{i=k}^{r-1}(1-p_i)\lambda_i}{\prod_{i=k}^{r}(s+\lambda_i)},$$

and similarly for $\boldsymbol{\beta}_k'(s)$, $\beta_k(s)$.

We use the usual tensor notation (see also Neuts [10], p.53)

$$(x_1, \dots, x_n)' \otimes (y_1, \dots, y_m)' = (x_1 y_1, \dots, x_n y_1, \dots, x_n y_m)'.$$

Finally, $\underline{\underline{\nu}}^+$, $\underline{\underline{\nu}}^0$, $\underline{\underline{\nu}}^-$, $\underline{\underline{\nu}}_0^0$, $\underline{\underline{\nu}}_1^-$, $\underline{\underline{\nu}}_0^+$ are transition rate matrices. We can express the transition rate matrices in terms of the matrices A_0, A_1, B_0, B_1 using tensor notation. For example

$$\underline{\underline{\nu}}^0 = -I \otimes A_0 - B_0 \otimes I.$$

3. The queue length distribution in the transient domain

The system $MGE_L/MGE_M/1$ is an instance of the homogeneous row-continuous Markov chain with a single boundary (Keilson and Zachmann [8]). We analyze the homogeneous part of the Markovian dynamics using the separation of variables technique combined with tensor algebra. Then we analyze the compensation part (i.e. the boundary condition plus the initial condition) using linear algebra. In this way we succeed in finding a closed form expression of the Laplace transform of the queue length distribution. We assume that the system is initially empty. Although our approach can be in principle applied even in the case of arbitrary initial conditions, the algebra required makes the explicit derivation very hard.

THE HOMOGENEOUS PART

Using the notation of section 2, we first write the Chapman–Kolmogorov forward equation for $n > 1$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{P}_n'(t) = \boldsymbol{P}_{n-1}'(t)\underline{\underline{\nu}}^+ + \boldsymbol{P}_n'(t)\underline{\underline{\nu}}^0 + \boldsymbol{P}_{n+1}'(t)\underline{\underline{\nu}}^-.$$

By taking the Laplace transform and using the assumption that $\boldsymbol{P}_n(0) = \boldsymbol{0}$ (for $n \geqslant 1$) we obtain:

$$s\boldsymbol{\pi}_n'(s) = \boldsymbol{\pi}_{n-1}'(s)\underline{\underline{\nu}}^+ + \boldsymbol{\pi}_n'(s)\underline{\underline{\nu}}^0 + \boldsymbol{\pi}_{n+1}'(s)\underline{\underline{\nu}}^-. \tag{1}$$

THE COMPENSATION PART

Similarly, the Chapman–Kolmogorov forward equation for $n = 0$ and $n = 1$ is:

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{P}_0'(t) = \boldsymbol{P}_0'(t)\underline{\underline{\nu}}_0^0 + \boldsymbol{P}_1'(t)\underline{\underline{\nu}}_1^-,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{P}_1'(t) = \boldsymbol{P}_0'(t)\underline{\underline{\nu}}_0^+ + \boldsymbol{P}_1'(t)\underline{\underline{\nu}}^0 + \boldsymbol{P}_2'(t)\underline{\underline{\nu}}^-,$$

and therefore the Laplace transform of the equations for $n = 0, 1$ is:

$$s\boldsymbol{\pi}_0'(s) = \boldsymbol{P}_0'(0) + \boldsymbol{\pi}_0'(s)\underline{\underline{\nu}}_0^0 + \boldsymbol{\pi}_1'(s)\underline{\underline{\nu}}_1^-, \tag{2}$$

$$s\boldsymbol{\pi}_1(s)=\boldsymbol{\pi}_0'(s)\underline{\nu}_0^+ +\boldsymbol{\pi}_1'(s)\underline{\nu}^0+\boldsymbol{\pi}_2'(s)\underline{\nu}^-. \tag{3}$$

Our general strategy for analyzing these equations is the following. We first find the general solution of (1) for $n>1$. The solution of the Laplace transform $\boldsymbol{\pi}_n(s)$ is then written as a linear combination of M geometric terms. The only unknowns are the M constants that depend on s (the coefficients D_r, $r=1,\dots,M$ below). From (2) we find $\boldsymbol{\pi}_0(s)$ as a function of the M coefficients D_r. Finally we use (3) to find a linear system of M equations with M unknowns. We exploit the particular structure of the linear system to find a closed form solution for the unknowns D_r, $r=1,\dots,M$. As a result, we find explicit closed form expressions for the Laplace transform of the queue length distribution.

The advantage of our expressions is that they can be numerically inverted in real time. In principle this approach works for arbitrary initial conditions. We were able to find closed form expressions only in the case in which the system is initially empty, i.e. $\boldsymbol{P}_n(0)=0$ for $n\geqslant 1$. We can now prove our first result.

THEOREM 1

Under the assumption that the system is initially empty, i.e. the only nonzero initial probabilities are $P_{0,k}(0)$ for all k, and $\rho<1$ the transform of the queue length distribution has the following form:

$$\boldsymbol{\pi}_n(s)=\sum_{r=1}^{M} D_r\boldsymbol{\beta}_1(x_r(s))\otimes\boldsymbol{\alpha}_1(s-x_r(s))\big(\alpha_1(s-x_r(s))\big)^{n-1}\quad\{n\geqslant 1\},$$

$$\boldsymbol{\pi}_0(s)=\sum_{k=1}^{L}P_{0,k}(0)\boldsymbol{\alpha}_k(s)+\sum_{r=1}^{M}D_r\frac{1}{x_r(s)}\beta_1(x_r(s))\big(\boldsymbol{\alpha}_1(s-x_r(s))-\boldsymbol{\alpha}_1(s)\big),$$

where

$$D_r=\frac{\sum_{k=1}^{L}P_{0,k}(0)\alpha_k(s)}{1-\alpha_1(s)}\,\frac{(-1)^M\beta_1^M(0)}{\beta_1^M(x_r(s))}x_r(s)\prod_{\substack{k=1\\k\neq r}}^{M}\frac{x_k(s)}{x_r(s)-x_k(s)},$$

and $x=x_r(s)$ (for $r=1,\dots,M$) are the M roots of the equation:

$$\begin{cases}\alpha_1(s-x)\beta_1(x)=1,\\ \mathscr{R}(x)<0\ (\text{i.e.}\,\|\alpha_1(s-x)\|<1)\text{ for }\mathscr{R}(s)>0.\end{cases}$$

Proof

We first find the general solution of (1). Equations (1) are partial difference equations with constant coefficients. Following the separation of variables technique we assume that a general solution $\widetilde{\pi_n}(s)$ of the difference eq. (1) is:

$$\widetilde{\pi_{n,i,j}}(s)=\phi_i(s)\psi_j(s)(w(s))^{n-1},$$

which can be written in tensor notation as follows:

$$\widetilde{\boldsymbol{\pi}_n}(s)=\boldsymbol{\phi}(s)\otimes\boldsymbol{\psi}(s)(w(s))^{n-1},$$

with

$$\boldsymbol{\phi}(s)=\begin{pmatrix}\phi_1(s)\\ \vdots\\ \phi_L(s)\end{pmatrix},\quad \boldsymbol{\psi}(s)=\begin{pmatrix}\psi_1(s)\\ \vdots\\ \psi_M(s)\end{pmatrix}.$$

We substitute the form of $\widetilde{\pi_n}(s)$ into (1) and using tensor notation we obtain:

$$\begin{aligned}&s\boldsymbol{\phi}'(s)\otimes\boldsymbol{\psi}'(s)(w(s))^{n-1}\\ &\quad=-\boldsymbol{\phi}'(s)A_1\otimes\boldsymbol{\psi}'(s)(w(s))^{n-2}-\boldsymbol{\phi}'(s)A_0\otimes\boldsymbol{\psi}'(s)(w(s))^{n-1}\\ &\qquad-\boldsymbol{\phi}'(s)\otimes\boldsymbol{\psi}'(s)B_0(w(s))^{n-1}-\boldsymbol{\phi}'(s)\otimes\boldsymbol{\psi}'(s)B_1(w(s))^{n},\end{aligned}$$

which by collecting terms can be written as

$$\begin{aligned}&s\boldsymbol{\phi}'(s)\otimes\boldsymbol{\psi}'(s)+\boldsymbol{\phi}'(s)\left(A_0+\frac{1}{w(s)}A_1\right)\otimes\boldsymbol{\psi}'(s)+\boldsymbol{\phi}'(s)\\ &\quad\otimes\boldsymbol{\psi}'(s)\left(B_0+w(s)B_1\right)=0.\end{aligned}$$

Our goal is to find one solution that satisfies eqs. (1), (2) and (3). Since the solution of (1), (2) and (3) is unique, our solution will indeed be this unique solution. For this reason, using the standard separation of variables arguments (see Bertsimas [2]) in tensor notation we require that $\boldsymbol{\phi}'(s)$ is a row eigenvector of the matrix $(A_0+(1/w(s))A_1)$ with an eigenvalue $-y(s)$ and $\boldsymbol{\psi}'(s)$ is a row eigenvector of the matrix $(B_0+w(s)B_1)$ with an eigenvalue $-x(s)$. As a result,

$$(s-x(s)-y(s))[\boldsymbol{\phi}'(s)\otimes\boldsymbol{\psi}'(s)]=0,$$

and therefore

$$s=x(s)+y(s). \tag{4}$$

In the following claim we establish the relation among $w(s)$, $y(s)$ and $x(s)$, by computing the characteristic polynomials of the matrices $(A_0+(1/w(s))A_1)$ and $(B_0+w(s)B_1)$.

CLAIM 1

$$\alpha_1(y(s))=w(s) \text{ and } w(s)\beta_1(x(s))=1.$$

Proof

Since $-x(s)$ is an eigenvalue of $(B_0+w(s)B_1)$, it satisfies the following characteristic equation:

$$\det[Ix(s)+B_0+w(s)B_1]=0,$$

that is

$$\det[Ix(s)+B_0]\det\left[I+w(s)(Ix(s)+B_0)^{-1}B_1\right]=0.$$

Since the matrix B_0 has full rank, $\det(Ix(s)+B_0) = \text{Denominator}[\beta_1(x(s))] = \prod_{r=1}^{M}(\mu_r + x(s)) \neq 0$. Also for any rank 1 matrix B, $\det(I+B) = 1 + \text{trace}(B)$. Thus, since B_1 has rank 1

$$1 + w(s)\text{trace}\Big((Ix(s)+B_0)^{-1}B_1\Big) = 0.$$

But $\beta_1(x(s)) = -\text{trace}((Ix(s)+B_0)^{-1}B_1)$ and therefore

$$w(s)\beta_1(x(s)) = 1.$$

Using exactly the same methodology we establish that

$$\alpha_1(y(s)) = w(s). \quad \square$$

We now compute in the following claim the eigenvectors $\boldsymbol{\psi}(s)$ and $\boldsymbol{\phi}(s)$ in closed form.

CLAIM 2

The vector $\boldsymbol{\alpha}_1(y(s))$ is a row eigenvector of the matrix $(A_0 + (1/w(s))A_1)$ and the vector $\boldsymbol{\beta}_1(x(s))$ is a row eigenvector of the matrix $(B_0 + w(s)B_1)$. Therefore we can choose $\boldsymbol{\phi}(s) = \boldsymbol{\alpha}_1(y(s))$ and $\boldsymbol{\psi}(s) = \boldsymbol{\beta}_1(x(s))$.

Proof

We prove the claim for $\boldsymbol{\beta}_1(x(s))$. The case for $\boldsymbol{\alpha}_1(y(s))$ is similar. Since $\boldsymbol{\beta}_1'(x(s)) = \boldsymbol{e}_1'(Ix(s)+B_0)^{-1}$ we have that

$$\begin{aligned}\boldsymbol{\beta}_1'(x(s))(B_0 + w(s)B_1) &= \boldsymbol{e}_1'(Ix(s)+B_0)^{-1}(B_0 + w(s)B_1)\\ &= \boldsymbol{e}_1'(Ix(s)+B_0)^{-1}\big(-Ix(s) + (Ix(s)+B_0)\\ &\quad + w(s)B_1\big)\\ &= -x(s)\boldsymbol{\beta}_1'(x(s)) + \boldsymbol{e}_1' - w(s)\beta_1(x(s))\boldsymbol{e}_1',\end{aligned}$$

since $\boldsymbol{\beta}_1'(s)B_1 = -\beta_1(s)\boldsymbol{e}_1'$. But from claim 1, $w(s)\beta(x(s)) = 1$ and thus

$$\boldsymbol{\beta}_1'(x(s))(B_0 + w(s)B_1) = -x(s)\boldsymbol{\beta}_1'(x(s)). \quad \square$$

As a result of (4) and claim 1, and since we are looking for roots $w(s) = \alpha_1(y(s)) = \alpha_1(s - x(s))$ inside the unit circle, so that the solution is stationary, the general solution of eq. (1) is

$$\widetilde{\boldsymbol{\pi}_n}(s) = \boldsymbol{\beta}_1(x(s)) \otimes \boldsymbol{\alpha}_1(s - x(s))(\alpha_1(s - x(s)))^{n-1}, \tag{5}$$

where, because of (4) and claim 1, $x = x(s)$ satisfy the equations

$$\begin{cases}\alpha_1(s-x)\beta_1(x) = 1,\\ \mathscr{R}(x) < 0(\text{i.e.} \parallel \alpha_1(s-x) \parallel < 1) \text{ for } \mathscr{R}(s) > 0.\end{cases} \tag{6}$$

In the following claim we investigate the number of roots of (6).

CLAIM 3

For $\rho < 1$ eq. (6) has M roots $x_r(s)$, $r = 1, \ldots, M$.

Proof

This can be easily established from an application of Rouche's theorem in the domain $\mathscr{R}(x) < 0$. For a very similar application of Rouche's theorem see Bertsimas [2]. The same result can be established from matrix geometric considerations by noticing that the roots $\alpha_1(s - x_r(s))$, $r = 1, \ldots, M$ are the M eigenvalues of the matrix $R(s)$ in Ramaswami [12]. $\square$

Assuming that the M roots of (6) are distinct we can now write an explicit expression for $\boldsymbol{\pi}_n(s)$, $n \geqslant 1$ by taking linear combinations of the general solution form. Indeed, there are coefficients D_r, $r = 1, \ldots, M$ such that

$$\boldsymbol{\pi}_n(s) = \sum_{r=1}^{M} D_r \boldsymbol{\beta}_1(x_r(s)) \otimes \boldsymbol{\alpha}_1(s - x_r(s))\big(\alpha_1(s - x_r(s))\big)^{n-1} \quad \{n \geqslant 1\}. \tag{7}$$

Remark: The distinctness assumption

This distinctness assumption, is very convenient in order to find an explicit expression for $\boldsymbol{\pi}_n(s)$, $n \geqslant 1$, but it is not critical however. The algebraic theory of rational functions guarantees that if there are multiple roots, we can take the limit of (7) as $x_r(s) \to x_k(s)$ for some r, k. In other words, we first solve the problem assuming that the roots are distinct and at the final stage we show the results are independent of this assumption. In fact, our final expressions for the queue length and the busy period distributions are simple *symmetric* functions of these roots. So, finding the limit in the case where there are multiple roots is an easy task.

The remaining unknowns are the coefficients $D_r (r = 1, \ldots, M)$ and $\boldsymbol{\pi}_0(s)$. In the following claim we express $\boldsymbol{\pi}_0(s)$ as a linear combination of the D_r's.

CLAIM 4

$$\boldsymbol{\pi}_0(s) = \sum_{k=1}^{L} P_{0,k}(0)\boldsymbol{\alpha}_k(s) + \sum_{r=1}^{M} D_r \frac{1}{x_r(s)} \beta_1(x_r(s))\big(\boldsymbol{\alpha}_1(s - x_r(s)) - \boldsymbol{\alpha}_1(s)\big). \tag{8}$$

Proof

We substitute (7) into the eq. (2) and we obtain

$$s\boldsymbol{\pi}_0'(s) = \boldsymbol{P}_0'(0) - \boldsymbol{\pi}_0'(s) A_0 - \sum_{r=1}^{M} D_r\big(\boldsymbol{\beta}_1(x_r(s)) B_1 \boldsymbol{e}_1\big)\boldsymbol{\alpha}_1'(s - x_r(s)).$$

Thus

$$\begin{aligned}\boldsymbol{\pi}_0'(s) &= \boldsymbol{P}_0'(0)(Is + A_0)^{-1} \\ &\quad + \sum_{r=1}^{M} D_r \beta_1(x_r(s)) \boldsymbol{e}_1'(I(s - x_r(s)) + A_0)^{-1}(Is + A_0)^{-1} \\ &= \sum_{k=1}^{L} P_{0,k}(0) \boldsymbol{\alpha}_k(s) \\ &\quad + \sum_{r=1}^{M} D_r \frac{1}{x_r(s)} \beta_1(x_r(s))(\boldsymbol{\alpha}_1(s - x_r(s)) - \boldsymbol{\alpha}_1(s)). \quad \square\end{aligned}$$

The next step in our approach is to find the coefficients D_r $(r = 1, \ldots, M)$. In the following claim we establish the equations from which the coefficients D_r are computed.

CLAIM 5

For all $k = 1, \ldots, M$:

$$\sum_{r=1}^{M} D_r \frac{1}{x_r(s)} \frac{\beta_1^k(x_r(s))}{\beta_1^k(0)} = -\frac{\sum_{k=1}^{L} P_{0,k}(0) \alpha_k(s)}{1 - \alpha_1(s)}. \tag{9}$$

Proof

Using (3) and (7) we easily obtain that

$$s\boldsymbol{\pi}_1'(s) - \boldsymbol{\pi}_1'(s)\underline{\underline{\nu}}^0 - \boldsymbol{\pi}_2'(s)\underline{\underline{\nu}}^- = \sum_{r=1}^{M} D_r \boldsymbol{\beta}_1'(x_r(s)).$$

From (3)

$$\sum_{r=1}^{M} D_r \boldsymbol{\beta}_1(x_r(s)) = -(\boldsymbol{\pi}_0'(s) A_1 \boldsymbol{e}_1) \boldsymbol{e}_1. \tag{10}$$

Using (8) and since $\boldsymbol{\alpha}_1'(s)(-A_1\boldsymbol{e}_1) = \alpha_1(s)$ we obtain that

$$\begin{aligned}-(\boldsymbol{\pi}_0'(s) A_1 \boldsymbol{e}_1) &= \boldsymbol{P}_0'(0)(Is + A_0)^{-1}(-A_1\boldsymbol{e}_1) \\ &\quad + \sum_{r=1}^{M} D_r \frac{1}{x_r(s)} \beta_1(x_r(s))(\alpha_1(s - x_r(s)) - \alpha_1(s)) \\ &= \alpha^*(s) + \sum_{r=1}^{M} D_r \frac{1 - \alpha_1(s)\beta_1(x_r(s))}{x_r(s)},\end{aligned}$$

where we defined $\alpha^*(s) = \boldsymbol{P}_0'(0)(Is + A_0)^{-1}(-A_1\boldsymbol{e}_1)$. We substitute this equation into (10) and obtain

$$\alpha^*(s)\boldsymbol{e}_1' = \sum_{r=1}^{M} D_r\left\{\boldsymbol{\beta}_1'(x_r(s)) - \frac{1-\alpha_1(s)\beta_1(x_r(s))}{x_r(s)}\boldsymbol{e}_1'\right\}$$

$$= \sum_{r=1}^{M} D_r\boldsymbol{\beta}_1'(x_r(s))\left\{I - \frac{Ix_r(s) + B_0}{x_r(s)} - \alpha_1(s)\frac{B_1}{x_r(s)}\right\}.$$

Since $\boldsymbol{\beta}_1'(s) = \boldsymbol{e}_1'(Is + B_0)^{-1}$ and $\beta_1(s)\boldsymbol{e}_1' = -\boldsymbol{\beta}_1'(s)B_1$, then

$$\alpha^*(s)\boldsymbol{e}_1' = \sum_{r=1}^{M} D_r\frac{1}{x_r(s)}\boldsymbol{\beta}_1'(x_r(s))\{Ix_r(s) - (Ix_r(s) + B_0) - \alpha_1(s)B_1\}$$

$$= -\sum_{r=1}^{M} D_r\frac{1}{x_r(s)}\boldsymbol{\beta}_1'(x_r(s))(B_0 + \alpha_1(s)B_1).$$

But $\boldsymbol{e}_1'(Is + B_0 + WB_1)^{-1} = \boldsymbol{\beta}_1'(s)/(1 - W\beta_1(s))$ and hence $\boldsymbol{e}_1'(B_0 + \alpha_1(s)B_1)^{-1} = \boldsymbol{\beta}_1'(0)/(1 - \alpha_1(s))$. As a result,

$$\sum_{r=1}^{M} D_r\frac{1}{x_r(s)}\boldsymbol{\beta}_1'(x_r(s)) = -\frac{\alpha^*(s)}{1-\alpha_1(s)}\boldsymbol{\beta}_1'(0)$$

$$= -\frac{\sum_{k=1}^{L}P_{0,k}(0)\alpha_k(s)}{1-\alpha_1(s)}\boldsymbol{\beta}_1'(0). \tag{11}$$

Therefore, for all $k = 1, \dots, M$ the coefficients D_r satisfy the linear system (9). □

We now solve the linear system (9) in closed form.

CLAIM 6

The linear system (9) has the following solution: For all $r = 1, \dots, M$:

$$D_r = \frac{\sum_{k=1}^{L}P_{0,k}(0)\alpha_k(s)}{1-\alpha_1(s)}\frac{(-1)^M\beta_1^M(0)}{\beta_1^M(x_r(s))}x_r(s)\prod_{\substack{k=1\\k\neq r}}^{M}\frac{x_k(s)}{x_r(s) - x_k(s)}.$$

Proof

Let

$$C_r = \frac{1-\alpha_1(s)}{\sum_{k=1}^{L}P_{0,k}(0)\alpha_k(s)}\frac{\beta_1^M(x_r(s))}{\beta_1^M(0)x_r(s)}D_r.$$

Since for all $k = 1, \dots, M$

$$\frac{\beta_1^k(s)}{\beta_1^k(0)}\frac{\beta_{k+1}^M(s)}{\beta_{k+1}^M(0)} = \frac{\beta_1^M(s)}{\beta_1^M(0)},$$

we obtain that for all $k = 1, \ldots, M$,

$$\sum_{r=1}^{M} C_r \frac{\beta_{k+1}^M(0)}{\beta_{k+1}^M(x_r(s))} = 1,$$

i.e.

$$\sum_{r=1}^{M} C_r \prod_{n=k+1}^{M} \left(1 + \frac{x_r(s)}{\mu_n}\right) = 1.$$

We expand the above equation as a polynomial of $x_r(s)$ and obtain

$$\sum_{r=1}^{M} \sum_{n=0}^{M-k} C_r \sigma_{k,n} (x_r(s))^n = 1,$$

where $\sigma_{k,n}$ are the coefficients in the expansion. We express this equation in matrix form and we obtain

$$\underbrace{\begin{bmatrix} 1 & & \sigma_{1,0} \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} (x_1(s))^{M-1} & \ldots & (x_M(s))^{M-1} \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} C_1 \\ \vdots \\ C_M \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}_{1}.$$

Since by the definition of the $\sigma_{k,n}$

$$\begin{bmatrix} 1 & & \sigma_{1,0} \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{bmatrix} \begin{pmatrix} x^{M-1} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \prod_{n=1}^{M-1} \left(1 + \dfrac{x}{\mu_{n+1}}\right) \\ \vdots \\ 1 \end{pmatrix}.$$

By letting $x = 0$, we obtain that $S^{-1} \mathbf{1} = \mathbf{e}_M$ and thus,

$$\mathbf{C} = A^{-1} \mathbf{e}_M.$$

We now observe that the matrix A is a Vandermode matrix. Using Cramer's rule to solve the above linear system and exploiting the property that the determinant of a Vandermode matrix generated from $u_1, \ldots, u_M$ (denoted by $V(u_1, \ldots, u_M)$) is given by $\prod_{i<j}(u_i - u_j)$, we obtain that

$$C_r = \frac{V(x_1(s), \ldots, x_{r-1}(s), 0, x_{r+1}(s), \ldots, x_M(s))}{V(x_1(s), \ldots, x_M(s))} = \prod_{\substack{k=1 \\ k \neq r}}^{M} \frac{x_k(s)}{x_k(s) - x_r(s)}.$$

Therefore,

$$D_r = \frac{\sum_{k=1}^{L} P_{0,k}(0) \alpha_k(s)}{1 - \alpha_1(s)} \frac{(-1)^M \beta_1^M(0)}{\beta_1^M(x_r(s))} x_r(s) \prod_{\substack{k=1 \\ k \neq r}}^{M} \frac{x_k(s)}{x_r(s) - x_k(s)}. \quad \square$$

Having found explicit solutions for the remaining unknowns D_r we have explicit expressions for the $\pi_n(s)$, $n \geqslant 0$. The proof of theorem 1 is now complete. □

As an additional check of the algebra we compute the generating function

$$\Psi(s, z) = \pi_0'(s) \cdot \mathbf{1} + \sum_{n=1}^{\infty} z^n \pi_n'(s) \cdot \mathbf{1}.$$

Summing up (7) and (8) we obtain that

$$\begin{aligned}\Psi(s, z) = {} & \frac{1}{s} + \frac{\sum_{k=1}^{L} P_{0,k}(0)\alpha_k(s)}{1-\alpha_1(s)} \\ & \times \sum_{r=1}^{M} \frac{(-1)^M \beta_1^M(0)}{\beta_1^M(x_r(s))} \frac{1-z}{s-x_r(s)} \frac{\beta_1(x_r(s))-1}{1-z\alpha_1(s-x_r(s))} \\ & \times \prod_{\substack{k=1 \\ k \neq r}}^{M} \frac{x_k(s)}{x_r(s)-x_k(s)}.\end{aligned}$$

For $z = 1$ we obtain that $\Psi(s, 1) = 1/s$, which is the condition that the probabilities sum to one in the transform domain. Another interesting point is the fact that the generating function $\Psi(s, z)$ is symmetric with respect to the roots $x_r(s)$. This observation is nontrivial and it is established by using the Lagrange interpolation formula and the Chinese remainder theorem. Theorem 1 was proved under the assumption that the roots $x_r(s)$ are distinct and therefore all the formulae are valid only in this case. If, however, there are multiple roots (say $x_i(s) = x_j(s)$), the resulting formulae are simply the limit of the formulae given here as $x_i(s) \to x_j(s)$.

4. Busy period analysis

Ramaswami [12] has characterized the busy period of an $G/Ph/1$ queue using the matrix geometric approach. In this section we simplify his result considerably and succeed in deriving a very simple formula for the Laplace transform of the busy period distribution which is very suitable for numerical inversion in the time domain as we show in the next section.

Let B_P be the length of the busy period. We also define a new random variable $\Delta_{i,j}$, which we call the queue build up time. This random variable plays a critical role in the analysis. $\Delta_{i,j}$ is the time between two arrival epochs with the following properties. Immediately after the initial arrival epoch there are n customers in the system and the STC is in state i, while immediately after the final arrival epoch (there may be other arrivals in between) there are $n+1$ customers in the system and the STC is in state j. In addition, throughout the time $\Delta_{i,j}$ the number of customers never decreased below n. Notice that $\Delta_{i,j}$ is independent of n.

Let $R(t)$ be a matrix whose i, j element is the pdf $R_{i,j}(t) = \mathrm{d}Pr[\Delta_{i,j} \leqslant t]/\mathrm{d}t$ of the queue build up time. Let $\Gamma_{i,j}(s) = E[\mathrm{e}^{-s\Delta_{i,j}}]$ be the transform of $R_{i,j}(t)$ and $\Gamma(s)$ be the transform of $R(t)$. Finally let $\xi'(s)$ be a row eigenvector of $\Gamma(s)$ with an eigenvalue $u(s)$. We can now state and prove our basic result.

THEOREM 2

The Laplace transform $\sigma(s)$ of the busy period B_P is given by

$$\sigma(s) = E\left[\mathrm{e}^{-sB_P}\right] = 1 - (1 - \beta_1(s)) \frac{\prod_{k=1}^{M}(s+\mu_k)}{\prod_{r=1}^{M}(s - x_r(s))}, \tag{12}$$

where $x_r(s)$ are the M roots of the polynomial eq. (6).

Remark

(12) is a simple symmetric function of $x_r(s)$ and therefore the distinctness assumption is no longer necessary.

Proof

By focusing on the last customer arrived in the busy period we write the busy period dynamics.

$$\frac{\mathrm{d}}{\mathrm{d}t} Pr[B_P \leqslant t] = \left\{ b_1(t) \int_t^\infty a_1(t)\, \mathrm{d}t \right\} + \sum_{n=1}^{\infty} \sum_{k=1}^{M} R_{1,k}^{(n)}(t)$$
$$* \left\{ \left[b_k(t) * b_1^{(n-1)}(t) * \left(\sum_{r=1}^{M} b_1^r(t) q_r \mu_r \right) \right] \int_t^\infty a_1(t)\, \mathrm{d}t \right\}, \tag{13}$$

where the symbol "$*$" indicates convolution, the first term corresponds to the case in which the last customer in the busy period was the only one in the busy period and in the second term (the double summation) we condition on the number of customers n and the state k of the STC this customer has found when he entered. In this case, the busy period is the sum of two independent random variables: The time to build n customers in the queue (a convolution of n build up times $\Delta_{1,k}$) and the time to empty the system, since he is the last arriving customer in the busy period.

Our strategy is first to find $R(t)$ and then using (13) to find the transform of the busy period. We are thus naturally led to the dynamics of the queue build up times. Similarly to the first passage time analysis as in Keilson and Zachmann

[8], we write down dynamics of the queue build up time in matrix form by considering the last arrival during $\Delta_{i,j}$

$$R(t)=\begin{bmatrix} b_1^1(t) & \dots & b_1^M(t) \\ \vdots & \ddots & \vdots \\ 0 & \dots & b_M^M(t) \end{bmatrix} a_1(t)$$

$$+\sum_{n=1}^{\infty} R^{(n)}(t) * \left\{ \left[\begin{pmatrix} b_1(t) \\ \vdots \\ b_M(t) \end{pmatrix} * b_1^{(n-1)}(t) * \left(b_1^1(t) \dots b_1^M(t)\right) \right] a_1(t) \right\}$$

$$= B(t) a_1(t) + \sum_{n=1}^{\infty} R^{(n)}(t) * \left[F_n(t) a_1(t) \right],$$

where $B(t)$, $F_n(t)$ are the upper diagonal matrix appearing as the first term in the sum and $F_n(t)$ is the matrix composed of the three convolutions. By taking the Laplace transform of the above matrix equation and multiply both sides from the left with the eigenvector of $\Gamma(s)\xi'(s)$, we obtain:

$$u(s)\xi'(s) = \xi'(s)\mathscr{L}\left\{ \left[B(t) + \sum_{n=1}^{\infty} u^n(s) F_n(t) \right] a_1(t) \right\}. \tag{14}$$

But,

$$(Is + B_0 + uB_1)^{-1} = (Is + B_0)^{-1} + \frac{u}{1-u\beta_1(s)} \begin{bmatrix} \beta_1(s)\beta_1'(s) \\ \vdots \\ \beta_M(s)\beta_1'(s) \end{bmatrix},$$

since for every pair of matrices C of full rank and D of rank 1, $(C+D)^{-1} = C^{-1} - (C^{-1}DC^{-1})/(1+tr(C^{-1}D))$. By expressing this in real time we obtain

$$e^{-(B_0+uB_1)t} = \begin{bmatrix} b_1^1(t) & \dots & b_1^M(t) \\ \vdots & \ddots & \vdots \\ 0 & \dots & b_M^M(t) \end{bmatrix}$$

$$+ \sum_{n=1}^{\infty} u^n \begin{pmatrix} b_1(t) \\ \vdots \\ b_M(t) \end{pmatrix} * b_1^{(n-1)}(t) * \left(b_1^1(t) \dots b_1^M(t)\right).$$

As a result, (14) becomes

$$u(s)\xi'(s) = \xi'(s)\mathscr{L}\left\{ e^{-(B_0+u(s)B_1)t} a_1(t) \right\} = \xi'(s)\alpha_1(Is + B_0 + u(s)B_1). \tag{15}$$

Therefore, since $\alpha_1(s)$ is a rational function, $\xi(s)$ must be a row eigenvector of

$(B_0 + u(s)B_1)$. If $-z(s)$ is the corresponding eigenvalue, following the same technique as in claim 1 we have that

$$u(s)\beta_1(z(s)) = 1. \tag{16}$$

Since $\alpha_1(s)$ is a rational function of s, we get from (15) that

$$u(s) = \alpha_1(s - z(s)). \tag{17}$$

Comparing (16) and (17) we observe that $z(s)$ satisfies exactly the same equations as $x(s)$ (eqs. (6)) and therefore $z(s) = x(s)$, $u(s) = w(s)$ and $\boldsymbol{\xi}(s) = \boldsymbol{\beta}_1(x(s))$.

Having characterized the eigenvalues and eigenvectors of $\Gamma(s)$ we can spectrum decompose it under the distinctness assumption:

$$(\Gamma(s))^n = \sum_{r=1}^{M} (\alpha_1(s - x_r(s)))^n \boldsymbol{\phi}_r(s)\boldsymbol{\beta}_1'(x_r(s)), \tag{18}$$

where

$$\underline{\underline{\beta}}(s) = \begin{bmatrix} \boldsymbol{\beta}_1'(x_1(s)) \\ \vdots \\ \boldsymbol{\beta}_1'(x_M(s)) \end{bmatrix} = [\boldsymbol{\phi}_1(s) \dots \boldsymbol{\phi}_M(s)]^{-1}.$$

After the characterization of the $\Gamma(s)$ we take the Laplace transform of (13). Using (18) and after similar manipulations with the analysis of the queue build up time, we obtain

$$\sigma(s) = \sum_{r=1}^{M} \boldsymbol{e}_1' \cdot \boldsymbol{\phi}_r(s) \frac{1 - \alpha_1(s - x_r(s))}{s - x_r(s)} \boldsymbol{\beta}_1(x_r(s))$$

$$= \boldsymbol{e}_1' \left(\underline{\underline{\beta}}(s)\right)^{-1} \begin{pmatrix} \dfrac{\beta_1(x_1(s)) - 1}{s - x_1(s)} \\ \vdots \\ \dfrac{\beta_1(x_M(s)) - 1}{s - x_M(s)} \end{pmatrix}.$$

Since for any non singular matrix A we know that $\boldsymbol{x}'A^{-1}\boldsymbol{y} = 1 - \det(A - \boldsymbol{y}\boldsymbol{x}')/\det(A)$, we find that

$$\sigma(s) = 1 - \frac{1}{\det\left(\underline{\underline{\beta}}(s)\right)} \det\left(\underline{\underline{\beta}}(s) - \begin{bmatrix} \boldsymbol{e}_1' \dfrac{\beta_1(x_1(s)) - 1}{s - x_1(s)} \\ \vdots \\ \boldsymbol{e}_1' \dfrac{\beta_1(x_M(s)) - 1}{s - x_M(s)} \end{bmatrix}\right).$$

But for all $r = 1, \dots, M$

$$
\begin{aligned}
e_1' \frac{\beta_1(x_r(s)) - 1}{s - x_r(s)} &= \frac{1}{s - x_r(s)} \{ e_1' \beta_1(x_r(s)) - e_1' \} \\
&= \frac{1}{s - x_r(s)} e_1' \{ (Ix_r(s) + B_0)^{-1}(-B_1) - I \} \\
&= \frac{1}{s - x_r(s)} e_1' (Ix_r(s) + B_0)^{-1} \{ -B_1 - (Ix_r(s) + B_0) \} \\
&= \frac{1}{s - x_r(s)} \boldsymbol{\beta}_1(x_r(s)) \{ I(s - x_r(s)) - (Is + B_1 + B_0) \} \\
&= \boldsymbol{\beta}_1(x_r(s)) - \frac{1}{-x_r(s)} \boldsymbol{\beta}_1(x_r(s))(Is + B_1 + B_0).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sigma(s) &= 1 - \frac{\det\left(\underline{\underline{\beta}}(s) - \left\{ \underline{\underline{\beta}}(s) - \text{diagonal}\left(\frac{1}{s - x_1(s)}, \dots, \frac{1}{s - x_M(s)} \right) \underline{\underline{\beta}}(s)(Is + B_1 + B_0) \right\} \right)}{\det\left(\underline{\underline{\beta}}(s) \right)} \\
&= 1 - \det\left(\text{diagonal}\left(\frac{1}{s - x_1(s)}, \dots, \frac{1}{s - x_M(s)} \right) \right) \det(Is + B_1 + B_0) \\
&= 1 - \frac{\text{Denominator}\beta_1(s) - \text{Numerator}\beta_1(s)}{\prod_{r=1}^{M}(s - x_r(s))} \\
&= 1 - \frac{\left(\prod_{k=1}^{M-1}(1 - q_k)\mu_k \right)(1 - \beta_1(s))}{\beta_1^M(s) \prod_{r=1}^{M}(s - x_r(s))} \\
&= 1 - (1 - \beta_1(s)) \frac{\prod_{k=1}^{M}(s + \mu_k)}{\prod_{r=1}^{M}(s - x_r(s))}. \quad \square
\end{aligned}
$$

Although the analysis used some rather heavy machinery from linear algebra, it used direct probabilistic arguments by considering the dynamics of the system. The reward of this analysis is a very simple expression for the transform of the busy period distribution, which as we show in section 6 offers very important computational advantages. In addition, it is not hard to compute in closed form moments of the busy period distribution by repeated differentiation of the Laplace transform.

5. The waiting time distribution under FCFS

In this section we derive an expression for the conditional waiting time pdf given the arriving time. It is also possible to find the waiting time of the nth arriving time using the methods of this paper.

Our strategy for the analysis is the following; we first find the distribution of the number of customers in the system at an arrival epoch; conditioned on the number of customers found upon arrival, we then find the waiting time pdf and finally we find the (unconditioned) waiting time cdf.

In order to obtain a closed form expression in the transform domain, we make the assumption that the system is initially empty and futhermore, the initial probability distribution has a very special form.

ASSUMPTION 1

We assume that the initial probability vector $\boldsymbol{P}_0(0) = \lambda \boldsymbol{\alpha}_1(0)$.

In principle, this assumption is not necessary if we take a pure numerical approach for the solution. Without this assumption, however, it is not possible to obtain a closed form expression both in real time and in the transform region. In the next theorem we prove a critical consequence of assumption 1; the arrival process has already reached steady state from the beginning, i.e.,

$$P_{0,i}(t) = \Pr[R_a(t) = i] = \Pr[R_a(0) = i] = P_{0,i}(0).$$

PROPOSITION 1

If the initial condition satisfies assumption 1, the arrival time of the first customer is the forward recurrence time of the interarrival time, i.e. the residual life time of the renewal interval. Furthermore, $\boldsymbol{P}_0(0)$ is the stationary solution of the Kolmogorov equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P}'(t) = -\boldsymbol{P}'(t)(A_0 + A_1)$$
$$\boldsymbol{P}'(t) \cdot \boldsymbol{1} = 1,$$

that describes the arrival process.

Proof

The transform of the interarrival time T_a^* of the first customer is given by:

$$\alpha^*(s) = \mathrm{E}\left[\mathrm{e}^{sT_a^*}\right] = \boldsymbol{P}_0'(0) \cdot \begin{pmatrix} \alpha_1(s) \\ \vdots \\ \alpha_L(s) \end{pmatrix}.$$

Since $\boldsymbol{\alpha}_1'(0) = \boldsymbol{e}_1' A_0^{-1}$ and

$$\begin{pmatrix} \alpha_1(s) \\ \vdots \\ \alpha_L(s) \end{pmatrix} = (Is + A_0)^{-1}(-A_1 \boldsymbol{e}_1),$$

we find after simple algebraic manipulations the transform of the forward recurrence time under assumption 1 as follows:

$$\begin{aligned}\alpha^*(s) &= \lambda \boldsymbol{\alpha}_1'(0) \cdot (Is + A_0)^{-1}(-A_1 \boldsymbol{e}_1)\\ &= \lambda \boldsymbol{e}_1'(A_0)^{-1}(Is + A_0)^{-1}(-A_1 \boldsymbol{e}_1)\\ &= \lambda \boldsymbol{e}_1' \frac{1}{s}\left((A_0)^{-1} - (Is + A_0)^{-1}\right)(-A_1 \boldsymbol{e}_1)\\ &= \frac{\lambda}{s}(1 - \alpha_1(s)).\end{aligned}$$

To obtain the stationary distribution, we observe that $(A_0 + A_1)\boldsymbol{1} = \boldsymbol{0}$ and because of the structure of A_1, we know that $A_1 \boldsymbol{1} = A_1 \boldsymbol{e}_1$. Thus we find

$$\boldsymbol{1} = -A_0^{-1} A_1 \boldsymbol{e}_1, \tag{19}$$

or equivalently

$$\boldsymbol{1}\boldsymbol{e}_1' = -A_0^{-1} A_1.$$

We are now ready to prove that the stationary probability vector is proportional to $\boldsymbol{\alpha}_1(0)$; for this it suffices to show $\boldsymbol{\alpha}_1'(0)(A_0 + A_1) = \boldsymbol{0}'$. Since $\boldsymbol{\alpha}_1'(0) = \boldsymbol{e}_1' A_0^{-1}$, we have

$$\begin{aligned}&\boldsymbol{\alpha}_1'(0)(A_0 + A_1)\\ &\quad = \boldsymbol{e}_1' A_0^{-1}(A_0 + A_1)\\ &\quad = \boldsymbol{e}_1' + \boldsymbol{e}_1' A_0^{-1} A_1\\ &\quad = \boldsymbol{e}_1' - (\boldsymbol{e}_1' \cdot \boldsymbol{1})\boldsymbol{e}_1'\\ &\quad = \boldsymbol{e}_1' - \boldsymbol{e}_1'\\ &\quad = \boldsymbol{0}'.\end{aligned}$$

We complete the proof by showing $1/\lambda = \boldsymbol{\alpha}_1'(0) \cdot \boldsymbol{1}$. Utilizing (19) and the definition $\alpha_1(s) = -\boldsymbol{e}_1'(Is + A_0)^{-1} A_1 \boldsymbol{e}_1$ we obtain

$$\begin{aligned}\boldsymbol{\alpha}_1'(0) \cdot \boldsymbol{1} &= -\boldsymbol{\alpha}_1'(0) A_0^{-1} A_1 \boldsymbol{e}_1\\ &= -\boldsymbol{e}_1' A_0^{-1} A_0^{-1} A_1 \boldsymbol{e}_1\\ &= -\lim_{s \to 0} \boldsymbol{e}_1' A_0^{-1}(Is + A_0)^{-1} A_1 \boldsymbol{e}_1\\ &= \lim_{s \to 0} \frac{1}{s} \boldsymbol{e}_1'\left((Is + A_0)^{-1} - A_0^{-1}\right) A_1 \boldsymbol{e}_1\\ &= \lim_{s \to 0} \frac{\alpha_1(0) - \alpha_1(s)}{s}\\ &= \frac{1}{\lambda}.\end{aligned}$$

Therefore, we have proved that the ergodic solution to the above Kolmogorov equation is $\lambda \boldsymbol{\alpha}_1(0) = \boldsymbol{P}_0(0)$. □

A corollary of the theorem is that the expression for D_r in theorem 1 further simplifies to

$$D_r = \frac{\lambda}{s} \frac{(-1)^M \beta_1^M(0)}{\beta_1^M(x_r(s))} x_r(s) \prod_{\substack{k=1 \\ k \neq r}}^{M} \frac{x_k(s)}{x_r(s) - x_k(s)}, \tag{20}$$

since

$$\sum_{k=1}^{L} P_{0,k}(0) \alpha_k(s) = \alpha^*(s) = \frac{\lambda}{s}(1 - \alpha_1(s)).$$

We will next find the distribution of the number of customers in the system seen by an arriving customer. We define the event AAO = Arrival about to occur in $(\tau, \tau + \mathrm{d}\tau)$ and the pre-arrival probabilities: $P_{n,i}^-(\tau) = Pr[N(\tau) = n, R_s(\tau) = i \mid AAO]$. In the following proposition we find the pre-arrival probabilities.

PROPOSITION 2

Under assumption 1 the vector of the pre-arrival probabilities is

$$\boldsymbol{P}_n'^-(r) = \frac{1}{\lambda} \boldsymbol{P}_n'(r) \{(-A_1 \boldsymbol{e}_1) \otimes I\},$$

and its Laplace transform is

$$\pi_n^-(s) = \frac{1}{\lambda} \sum_{r=1}^{M} D_r \boldsymbol{\beta}_1(x_r(s)) (\alpha_1(s - x_r(s)))^n \quad \{n \geqslant 1\},$$

$$\pi_0^-(s) = \frac{1}{\lambda} \sum_{r=1}^{M} D_r \beta_1^1(x_r(s)).$$

Proof

$$P_{n,i}^-(\tau) = \Pr[N(\tau) = n,\ R_s(\tau) = i \mid AAO] = \frac{\Pr[N(\tau) = n \cap R_s(\tau) = i \cap AAO]}{\Pr[AAO]}$$

$$= \frac{\Pr\left[\cup_{l=1}^{k} (N(\tau) = n \cap R_s(\tau) = i \cap R_a(\tau) = l) \cap AAO\right]}{\Pr\left[\cup_{l=1}^{k} R_a(\tau) = l \cap AAO\right]}$$

$$= \frac{\Sigma_{l=1}^{k} \Pr[AAO \mid N(\tau) = n \cap R_s(\tau) = i \cap R_a(\tau) = l] \Pr[N(\tau) = n \cap R_s(\tau) = i \cap R_a(\tau) = l]}{\Sigma_{l=1}^{k} \Pr[AAO \mid R_a(\tau) = l] \Pr[R_a(\tau) = l]}.$$

But

$$\Pr[AAO \mid N(\tau)=n \cap R_s(\tau)=i \cap R_a(\tau)=l]$$
$$= \Pr[AAO \mid R_a(\tau)=l] = \lambda_l p_l \, \mathrm{d}\tau$$

and

$$\Pr[R_a(\tau)=l] = P_{0,l}(0) = \lambda \alpha_1^l(0),$$

from proposition 1. It is at this point that assumption 1 becomes critical. Without this assumption $\Pr[R_a(\tau)=l]$ would be a function of τ, while under assumption 1 it is independent of τ, and therefore it would not be possible to find a closed form formula for the transform of the pre-arrival distribution. Therefore, we obtain

$$P_{n,i}^-(\tau) = \frac{\sum_{l=1}^k \lambda_l p_l P_{n,l,i}(\tau)}{\sum_{l=1}^k \lambda \lambda_l p_l \alpha_1^l(0)},$$

since $\sum_{l=1}^k \lambda_l p_l \alpha_1^l(0) = \alpha_1(0) = 1$. Therefore, using vector–tensor notation we have:

$$\boldsymbol{P}_n'^-(\tau) = \frac{1}{\lambda} \boldsymbol{P}_n'(\tau)\{(-A_1 \boldsymbol{e}_1) \otimes I\}.$$

In the transform region, using

$$\boldsymbol{\pi}_n(s) = \sum_{r=1}^M D_r \boldsymbol{\beta}_1(x_r(s)) \otimes \boldsymbol{\alpha}_1(s - x_r(s))(\alpha_1(s - x_r(s)))^{n-1} \quad \{n \geqslant 1\},$$

we obtain (the derivation for $\pi_0^-(s)$ is similar)

$$\boldsymbol{\pi}_n^-(s) = \frac{1}{\lambda} \sum_{r=1}^M D_r \boldsymbol{\beta}_1(x_r(s))(\alpha_1(s - x_r(s)))^n \quad \{n \geqslant 1\},$$

$$\pi_0^-(s) = \frac{1}{\lambda} \sum_{r=1}^M D_r \beta_1^1(x_r(s)). \quad \square$$

We are now ready to prove the central theorem of this section.

THEOREM 3

Under assumption 1 the Laplace transform of the waiting time distribution under FCFS is

$$\int_0^\infty \mathrm{e}^{-s\tau} \Pr[W(\tau) \leqslant t] \, \mathrm{d}\tau = \frac{1}{s} + \sum_{r=1}^M \frac{(-1)^M \beta_1^M(0)}{s \beta_1^M(x_\tau(s))} \times \left(\prod_{\substack{k=1 \\ k \neq r}}^M \frac{x_k(s)}{x_r(s) - x_k(s)} \right) \mathrm{e}^{x_r(s)t}.$$

Proof

Given there are exactly n customers in the system including the customer just arrived and the STC is in stage i, then waiting time c.d.f. is:

$$\begin{cases} \int_0^t b_i(t) * b_1^{(n-2)}(t) * \sum_{j=1}^M b_1^j(t) q_j \mu_j \, \mathrm{d}t & \text{when } n \geqslant 3, \\ \int_0^t \sum_{j=1}^M b_i^j(t) q_j \mu_j \, \mathrm{d}t & \text{when } n = 2, \\ U(t) & \text{when } n = 1, \end{cases}$$

where $U(t)$ is unit step function. By conditioning on the state the arriving customer found the system, and using the expressions for the pre-arrival probabilities from proposition 2 we obtain:

$$\int_0^\infty \mathrm{e}^{-s\tau} \Pr[W(\tau) \leqslant t] \, \mathrm{d}\tau$$

$$= \frac{1}{\lambda} \sum_{r=1}^M D_r \beta_1'(x_r(s)) \left(e_1 U(t) + \int_0^t \left\{ \alpha_1(s - x_r(s)) \begin{bmatrix} b_1^1(t) & \dots & b_1^M(t) \\ \vdots & \ddots & \vdots \\ 0 & \dots & b_M^M(t) \end{bmatrix} \right. \right.$$

$$\left. + \sum_{n=1}^\infty \left(\alpha_1(s - x_r(s)) \right)^{n+1} \begin{pmatrix} b_1(t) \\ \vdots \\ b_M(t) \end{pmatrix} * b_1^{(n-1)}(t) * \left(b_1^1(t) \dots b_1^M(t) \right) \right\}$$

$$\left. \times \begin{pmatrix} q_1 \mu_1 \\ \vdots \\ \mu_M \end{pmatrix} \mathrm{d}t \right). \tag{21}$$

We have observed in the analysis of busy period from the previous section that

$$\mathrm{e}^{-(B_0 + uB_1)t} = \begin{bmatrix} b_1^1(t) & \dots & b_1^M(t) \\ \vdots & \ddots & \vdots \\ 0 & \dots & b_M^M(t) \end{bmatrix}$$

$$+ \sum_{n=1}^\infty u^n \begin{pmatrix} b_1(t) \\ \vdots \\ b_M(t) \end{pmatrix} * b_1^{(n-1)}(t) * \left(b_1^1(t) \dots b_1^M(t) \right).$$

Substituting this into (21) we obtain

$$\begin{aligned}\int_0^\infty & \mathrm{e}^{-s\tau}\Pr[W(\tau)\leqslant t]\ \mathrm{d}\tau \\ &= \frac{1}{\lambda}\sum_{r=1}^{M} D_r \boldsymbol{\beta}_1'(x_r(s)) \\ &\quad\times\left(\boldsymbol{e}_1 U(t) + \int_0^t \alpha_1(s-x_r(s))\ \mathrm{e}^{-(B_0+\alpha_1(s-x_r(s))B_1)t}(-B_1\boldsymbol{e}_1)\ \mathrm{d}t\right).\end{aligned}$$

Since $\boldsymbol{\beta}_1'(x_r(s))$ is a row eigenvector of $(B_0+\alpha_1(s-x_r(s))B_1)$ with eigenvalue $-x_r(s)$ and $\alpha_1(s-x_r(s))\boldsymbol{\beta}_1'(x_r(s))(-B_1\boldsymbol{e}_1)=1$, we obtain

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-s\tau}\Pr[W(\tau)\leqslant t]\ \mathrm{d}\tau &= \frac{1}{\lambda}\sum_{r=1}^{M} D_r\left(\boldsymbol{\beta}_1'(x_r(s))\ \boldsymbol{e}_1 U(t) + \int_0^t \mathrm{e}^{x_r(s)t}\ \mathrm{d}t\right) \\ &= \frac{1}{\lambda}\sum_{r=1}^{M} D_r U(t)\left(\boldsymbol{\beta}_1'(x_r(s))\boldsymbol{e}_1 + \frac{\mathrm{e}^{x_r(s)t}-1}{x_r(s)}\right) \\ &= \frac{1}{\lambda}\sum_{r=1}^{M} D_r U(t)\left(\boldsymbol{\beta}_1'(x_r(s))\left(I - \frac{Ix_r(s)+B_0}{x_r(s)}\right)\boldsymbol{e}_1\right. \\ &\qquad \left. + \frac{\mathrm{e}^{x_r(s)t}}{x_r(s)}\right) \\ &= \frac{1}{\lambda}\sum_{r=1}^{M} D_r U(t)\left(-\frac{1}{x_r(s)}\boldsymbol{\beta}_1'(x_r(s))B_0\boldsymbol{e}_1\right. \\ &\qquad \left. + \frac{\mathrm{e}^{x_r(s)t}}{x_r(s)}\right).\end{aligned}$$

Substituting expression (20) for D_r and using (11) we finally obtain

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-s\tau}\Pr[W(\tau)\leqslant t]\ \mathrm{d}\tau &= \frac{1}{s} + \sum_{r=1}^{M}\frac{(-1)^M\beta_1^M(0)}{s\beta_1^M(x_r(s))} \\ &\quad\times\left(\prod_{\substack{k=1\\k\neq r}}^{M}\frac{x_k(s)}{x_r(s)-x_k(s)}\right)\mathrm{e}^{x_r(s)t}. \quad\square\end{aligned}$$

A corollary from the previous theorem is that

$$\begin{aligned}\Phi(s,\omega) &= \int_0^\infty\int_0^\infty \mathrm{e}^{-s\tau-\omega t}\frac{\partial}{\partial t}\Pr[W(\tau)\leqslant t]\ \mathrm{d}t\ \mathrm{d}\tau \\ &= \frac{\beta_1^M(0)}{s\beta_1^M(\omega)}\prod_{r=1}^{M}\frac{x_r(s)}{x_r(s)-\omega}.\end{aligned}$$

Note that this expression is a symmetric function of the $x_r(s)$'s. In addition the quantity $\lim_{s\to 0} s\Phi(s, \omega)$ is the solution of the steady state Lindley equation for the $GI/G/1$ queue, i.e. the transform of the steady state waiting time distribution.

6. Numerical results

In the previous sections we have derived explicit expressions for the Laplace transforms of the queue length, the waiting time and the busy period distributions. In this section we will remove the "Laplacian curtain", by numerically inverting the Laplace transforms. The numerical inversion of the Laplace transform is a well studied but not completely solved problem in numerical analysis. In fact, Platzman, Ammons, and Bartholdi [11] show that the problem of numerically inverting the Laplace transform of a probability distribution is $\#P$-complete, that is, a hard computational problem.

Our overall algorithm is written using the software package of Mathematica, developed by Wolfram [13] and works as follows. We first compute the roots of the polynomial eqs. (6) for selected s values. For this purpose we use the build in functions of Mathematica to find all the roots of (6). We then use the algorithms of [11] and [7] to compute the inverse Laplace transform of the distributions under study. We used two algorithms to invert numerically the Laplace transforms:

(1) *The algorithm of Platzman et al. [11].*

This algorithm works for distributions that are defined over finite regions. We used this algorithm combined with fast Fourier transform for the inversion of the busy period distribution. Although the busy period takes values in the region $(0, \infty)$, in order to apply the algorithm we used the region $(0, E[B_P] + 3\sqrt{Var[B_P]})$ as the region on which the busy period is different from 0. For details of this algorithm the reader is referred to [11].

(2) *The algorithm by Hosono [7].*

Hosono [7] proposed an algorithm for inverting Laplace transforms which is quite robust and accurate. We used this algorithm for numerically inverting the busy period, the waiting time and the queue length distributions. The algorithm is not well-known in the western literature since it is primarily published in Japanese. We found however that it is a very robust algorithm. It satisfies all the necessary conditions which an ideal fast algorithm should satisfy, namely: It is easy to program and control the error. Moreover, it has small memory requirements, short computational times and can be used for a wide variety of problems. We briefly introduce the algorithm below.

Let $\phi(s)$ be the input function. Let $f(t)$ be the inverse Laplace transform of $\phi(s)$. We choose a precision p {significant digits} so that error of numerical inversion is less than $10^{-p+1}\,|\,f(t)\,|$. Let

$$F_n \triangleq \frac{(-1)^n \, \mathrm{e}^p}{t} \operatorname{Im}\left[\phi\left(\frac{p+\mathrm{i}\pi(n-0.5)}{t}\right)\right],$$

and

$$C_n \triangleq \left(\frac{1}{2}\right)^p \sum_{r=0}^{p-n+1} \binom{p}{r}.$$

The algorithm works as follows:

(a) Find k so that

$$\left|\sum_{r=0}^{p} \binom{p}{r} F_{k+r}\right| < \left(\frac{2}{\mathrm{e}^2}\right)^p.$$

(b) Evaluate $f(t)$:

$$f(t) \approx \sum_{n=1}^{k-1} F_n + \sum_{r=0}^{p-1} C_r F_{k+r}.$$

Hosono claims that this algorithm works when $f(t)$ is sufficiently smooth. In particular F_n needs to satisfy the following conditions:

(a) for sufficiently large n, $1/2 < |\,F_{n+1}/F_n\,| \leqslant 1$;

(b) when $n \to \infty$, F_n, ΔF_n, $\Delta^2 F_n$, ... converge monotonically to 0, where Δ^r denotes the rth difference.

It can be shown that the violation of these conditions results in the Gibbs phenomenon, which only appears at points of discontinuity of $f(t)$.

COMMENTS ON THE NUMERICAL RESULTS

All the computation was done in a Macintosh II, and all the program is written in Mathematica. For computing the transient queue length and the waiting time distribution, we assume that the arrival time of the first customer is the forward recurrence time of the interarrival distribution.

We tested our algorithm for various $MGE_L/MGE_M/1$ cases. From our preliminary experience we can say that the algorithms, in particular the algorithm by Hosono, are robust and run very fast. The largest example we ran was an $MGE_{20}/MGE_{20}/1$, which was solved by the algorithms without any difficulty. Unfortunately, we did not have any other numerical results to compare with except the ones for the $M/M/1$ queue, whose solution is known explicitly in terms of modified Bessel functions (see Gross and Harris [6], p.143). Indeed, for various values of ρ ranging from 0.1–0.99 the algorithm gave identical results with the exact known values. In example 1 below we compare the results of this

Table 1
The $M/M/1$ busy period CDF

t	Exact	P.A.B.	Hosono
0.	0.	0.	0.
0.94104	0.565744	0.529196	0.565744
2.07421	0.734767	0.723878	0.734767
3.20737	0.806977	0.811223	0.806977
4.34054	0.848456	0.854939	0.848456
5.47371	0.875836	0.883652	0.875836
6.60688	0.875836	0.904243	0.875836
7.74004	0.9102	0.919855	0.9102
8.87321	0.921766	0.932158	0.921766
10.0064	0.931074	0.942134	0.931074
11.1395	0.938727	0.9504	0.938727
12.2727	0.945129	0.957369	0.945129
13.4059	0.950558	0.963326	0.950558
14.539	0.955216	0.968478	0.955216
15.6722	0.959252	0.972977	0.959252
16.8054	0.962779	0.976937	0.962779
17.9386	0.965883	0.980449	0.965883
19.0717	0.968631	0.983583	0.968631
20.2049	0.97108	0.986393	0.97108
21.3381	0.973271	0.988927	0.973271
22.4712	0.975241	0.991219	0.975241
23.6044	0.977019	0.993303	0.977019
24.7376	0.97863	0.995202	0.97863
25.8707	0.980094	0.996939	0.980094
29.2702	0.983766	1.	0.983766

algorithm to the exact results for a particular $M/M/1$ queue. As an illustration of the algorithms we present in example 2 the solution of an $MGE_3/MGE_2/1$ queue.

We start our examples with an $M/M/1$ queue with traffic intensity $\rho = 0.75$. The interarrival rate is $\lambda = 1$ and service completion rate $\mu = 4/3$. In order to compare the accuracy of the two algorithms with the known solution we computed in table 1 the CDF of the busy period. The mean is $E[B_P] = 3$ and the coefficient of variation is $C_B^2 = 7$. The algorithm by Hosono gives identical results with the exact solution.

In fig. 2 we plot the first and second moments of the waiting time as a function of t. In fig. 3 we plot the waiting time distribution as a function of t. In figs. 4, 5 we plot the first and second moments and the distribution of the queue length as a function of t. In all cases the algorithm by Hosono gave identical results compared to the exact values.

Merely as an illustration of the algorithms we chose an $MGE_3/MGE_2/1$

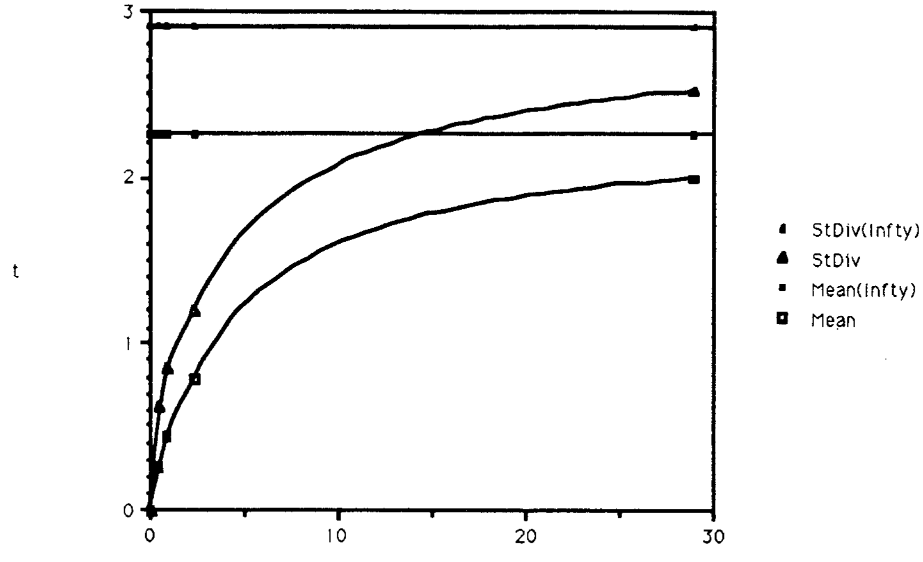

Fig. 2. The first and second moments of the waiting time of an $M/M/1$ queue as a function of time.

queue with the following distributions: The interarrival distribution has Laplace transform:

$$\alpha_1(s) = 0.5\frac{2}{2+s} + 0.5 \times 0.3\frac{2}{2+s}\,\frac{4}{4+s} + 0.5 \times 0.3\frac{2}{2+s}\,\frac{4}{4+s}\,\frac{6}{6+s},$$

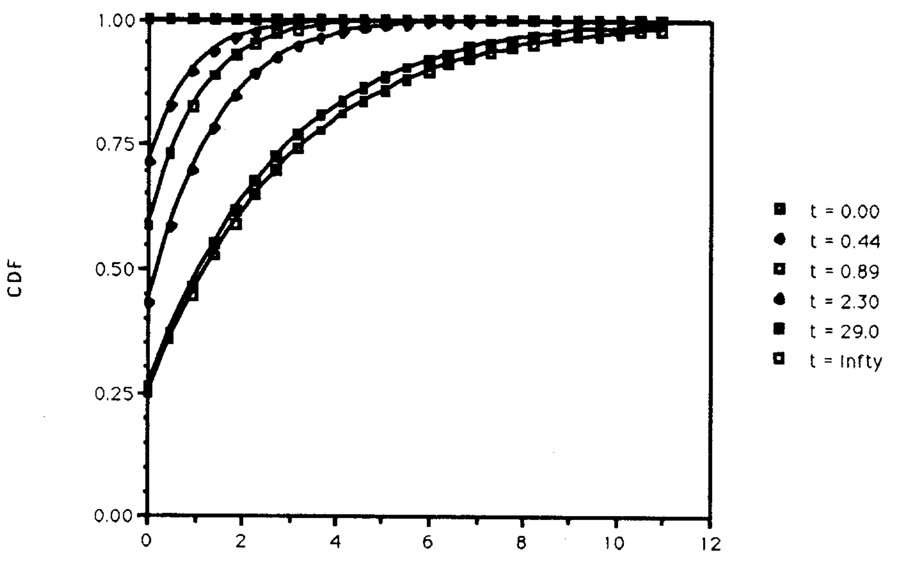

Fig. 3. The waiting time distribution of an $M/M/1$ queue as a function of time.

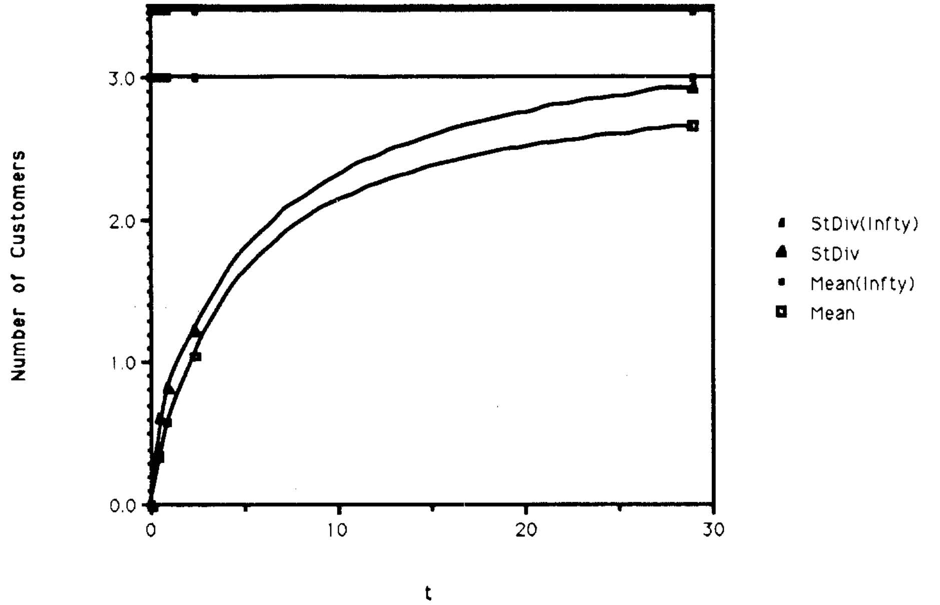

Fig. 4. The first and second moments of the queue length of an $M/M/1$ queue as a function of time.

with mean $E[T] = 0.683333$ and coefficient of variation is $C_T^2 = 0.70$. The service distribution has Laplace transform:

$$\beta_1(s) = 0.8\frac{5}{5+s} + 0.2\frac{5}{5+s}\,\frac{3}{3+s},$$

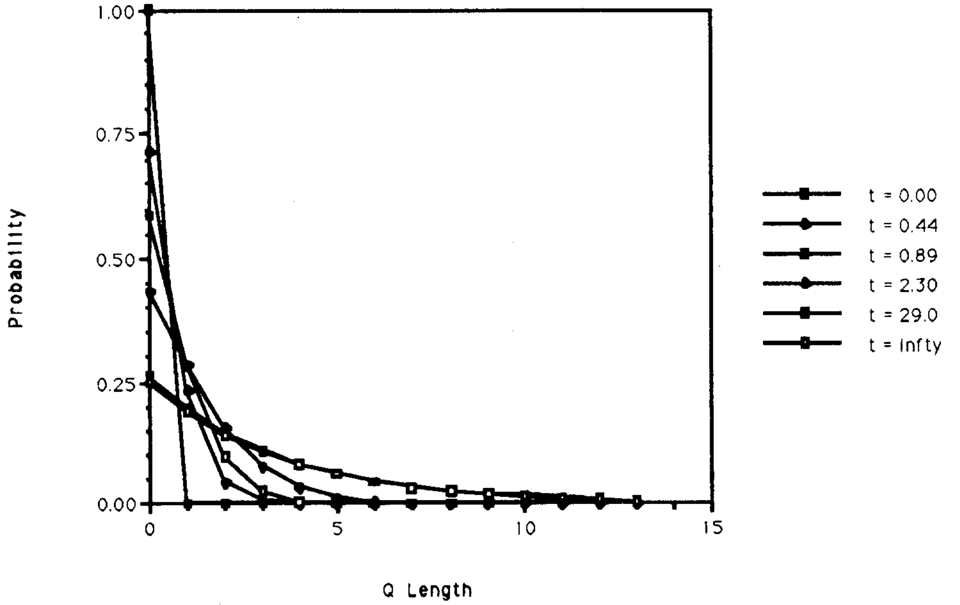

Fig. 5. The queue length distribution of an $M/M/1$ queue as a function of time.

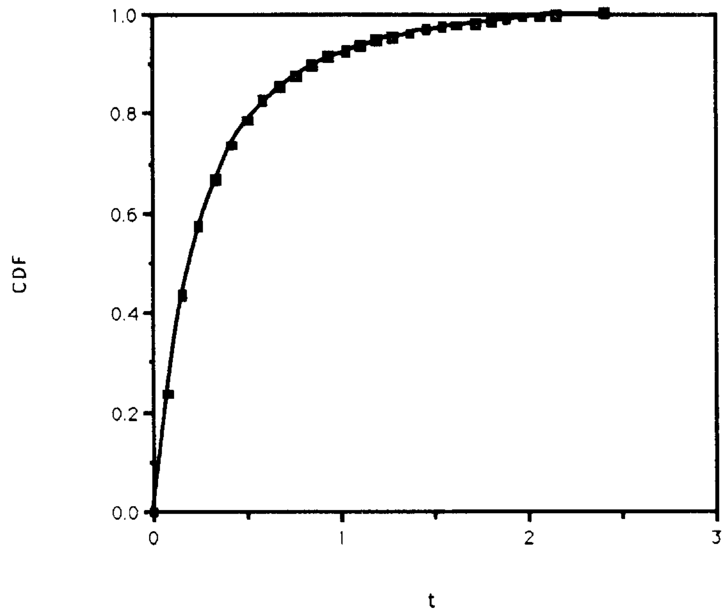

Fig. 6. The busy period CDF of an $MGE_3/MGE_2/1$ queue.

with mean $E[X] = 0.266667$ and coefficient of variation is $C_X^2 = 1.125$; thus the traffic intensity is $\rho = 0.39$.

By differentiating the transform of the busy period distribution we found that

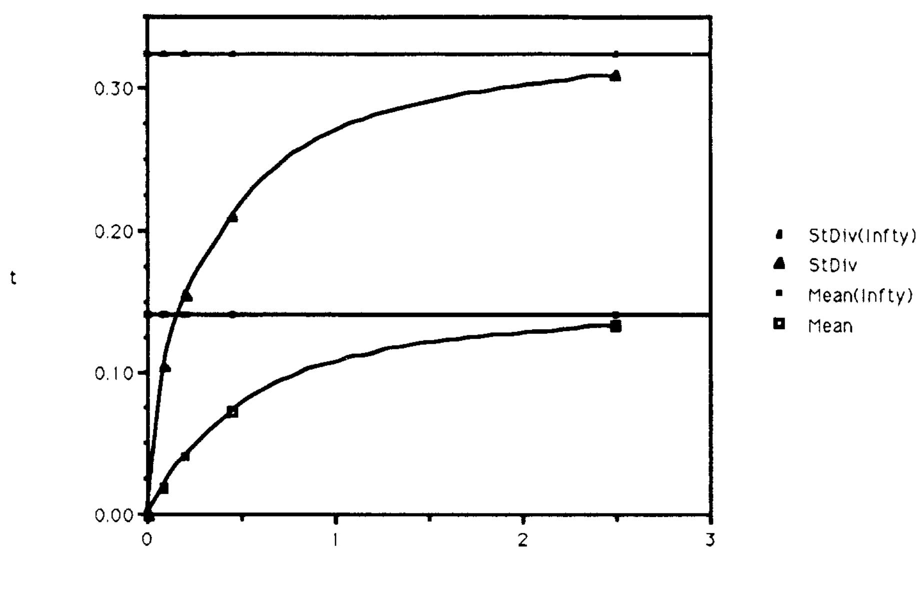

Fig. 7. The first and second moments of the waiting time of an $MGE_3/MGE_2/1$ queue as a function of time.

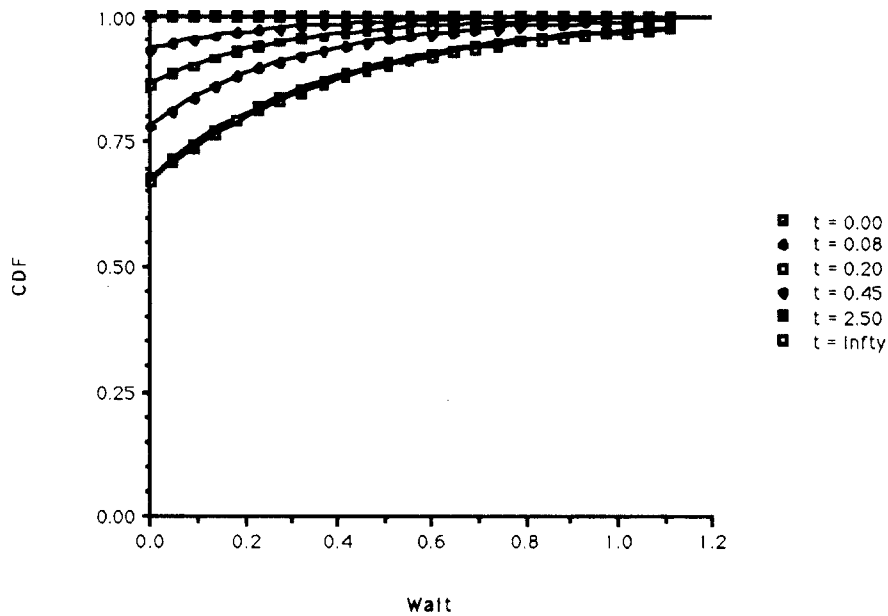

Fig. 8. The waiting time distribution of an $MGE_3/MGE_2/1$ queue as a function of time.

the mean of the busy period is $E[B_P] = 0.398444$ and the coefficient of variation is $C_B^2 = 2.327$. In fig. 6 we plot the busy period CDF, in fig. 7 we plot the first and second moments of the waiting time as a funtion of t. In fig. 8 we plot the

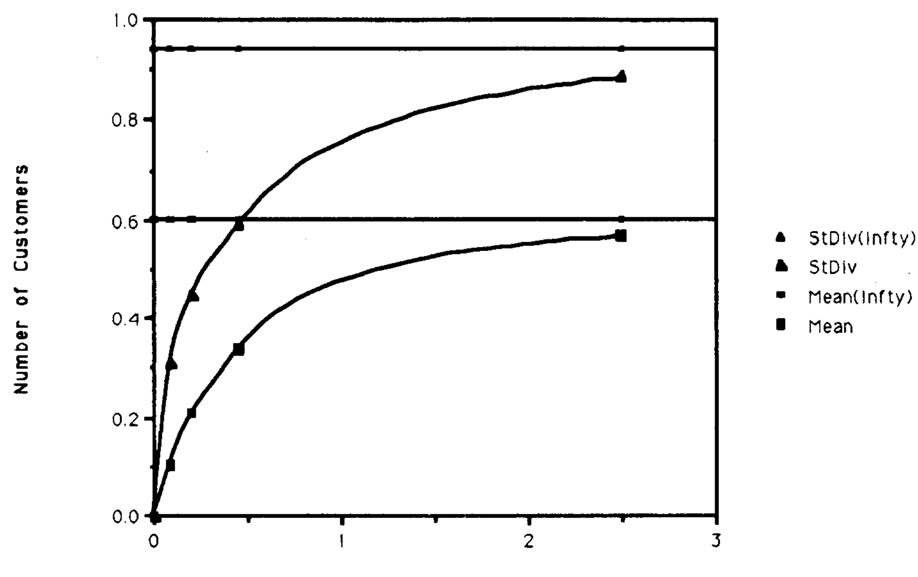

Fig. 9. The first and second moments of the queue length of an $MGE_3/MGE_2/1$ queue as a function of time.

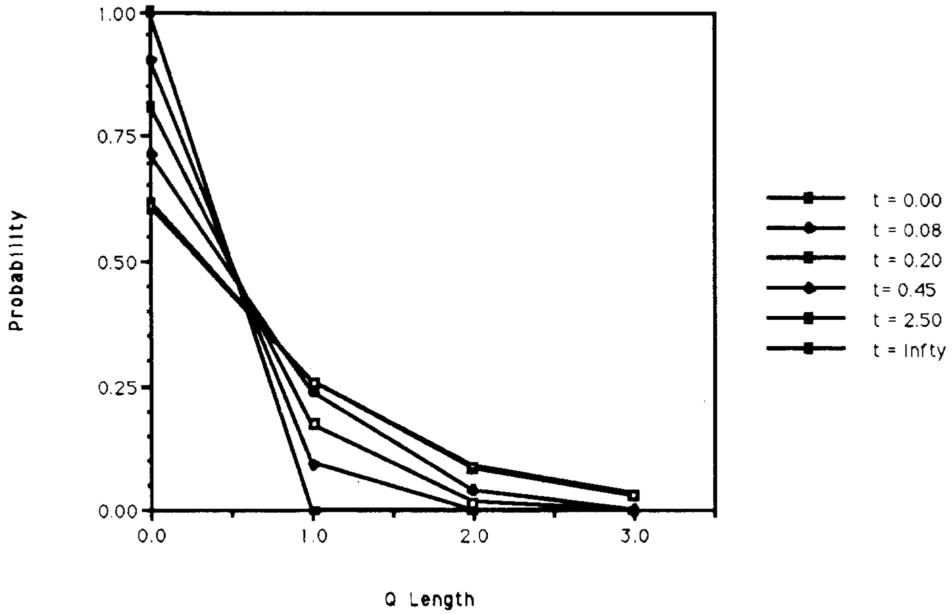

Fig. 10. The queue length distribution of $MGE_3/MGE_2/1$ queue as a function of time.

waiting time distribution as a function of t. In figs. 9 and 10 we plot the first and second moments and the distribution of the queue length as a function of t.

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